Combinatorics in Banach space theory (MIM UW 2014/15)

PROBLEMS (Part 3)

PROBLEM 3.1. Recall that for the (Figiel–Johnson version of the) original Tsirelson space $\mathcal{T}$ the norm on c_{00} is defined as the unique norm $\|\cdot\|$ satisfying the implicit formula:

$$\|x\| = \max\left\{\|x\|_\infty,\ \theta \sup_{m \leqslant E_1 < \ldots < E_m} \sum_{j=1}^{m} \|E_j x\|\right\},$$

where $\theta = \frac{1}{2}$. Show that in each of the two following cases:

- if we set $\theta = 1$ instead of $\theta = \frac{1}{2}$, or
- if we allow arbitrary decompositions $E_1 < \ldots < E_m$ instead of just those which are $\mathcal{S}$-admissible ($\mathcal{S} = \{A \subset \mathbb{N} \colon \#A \leqslant \min A\}$, the Schreier family),

after completion we obtain nothing but an isomorphic copy of ℓ_1.

PROBLEM 3.2. Recall that while proving the relation $\ell_1 \not\hookrightarrow \mathcal{T}$ we used the Bessaga–Pełczyński selection principle (combined with James' ℓ_1-distortion theorem) by saying that we may assume, towards a contradiction, that there exists a normalized *block* basic subsequence of the canonical basis $(e_n)_{n=1}^\infty$ of $\mathcal{T}$ which is equivalent (or even $(1+\varepsilon)$-equivalent) to the canonical basis of ℓ_1. Justify this statement by deriving the following corollary from the Bessaga–Pełczyński theorem: If X is a Banach space with a basis $(x_n)_{n=1}^\infty$, then every its infinite-dimensional subspace contains a basic sequence equivalent to a block basic subsequence of $(x_n)_{n=1}^\infty$.

Remark. Notice that our claim does not follow directly from the original version of the Bessaga–Pełczyński theorem, as opposed to case of ℓ_p's with $p > 1$. The difference is that in the latter case the canonical basis (and any its isomorphic copy) is weakly null which is not true for the canonical basis of ℓ_1 and hence the second requirement in the Bessaga–Pełczyński selection principle ($\lim_{n\to\infty} e_k^*(x_n) = 0$ for each k) is not automatically guaranteed.

PROBLEM 3.3. Let $\mathcal{M} \subset [\mathbb{N}]^{<\infty}$ be a compact family, closed under subsets and containing all singletons, and let $\theta \in (0,1)$. We define $W(\mathcal{M},\theta)$ to be smallest subset W of $\mathcal{T}(\mathcal{M},\theta)^*$ such that:

- $\pm e_k^* \in W$ for each $k \in \mathbb{N}$;
- W is closed under the $(\mathcal{M},\theta)$-operation, that is, if $f_1, \ldots, f_d \in W$ with $\mathrm{supp} f_1 < \ldots < \mathrm{supp} f_d$ being $\mathcal{M}$-admissible (which means that for some $\{m_1, \ldots, m_d\} \in \mathcal{M}$ we have $m_1 \leqslant \min \mathrm{supp} f_1 < m_2 \leqslant \min \mathrm{supp} f_2 < \ldots < m_d \leqslant \min \mathrm{supp} f_d$), then $\theta(f_1 + \ldots + f_d) \in W$.

Show that $W(\mathcal{M},\theta)$ is a norming subset of the unit dual ball of $\mathcal{T}(\mathcal{M},\theta)$.

Similarly, show that for any mixed Tsirelson space $\mathcal{T}[(\mathcal{M}_n,\theta)_{n=1}^\infty]$, where $\{\mathcal{M}_n\}_{n=1}^\infty$ are subfamilies of $[\mathbb{N}]^{<\infty}$ as above, $\{\theta_n\}_{n=1}^\infty \subset (0,1)$ and $\theta_n \to 0$, one obtains a norming subset $W[(\mathcal{M}_n,\theta_n)_{n=1}^\infty]$ of the unit dual ball by taking the smallest set containing all functionals of the form $\pm e_k^*$ and closed under all $(\mathcal{M}_n,\theta_n)$-operations ($n \in \mathbb{N}$).

Finally, show that for each norming set W obtained as above the corresponding unit dual ball is $\overline{\mathrm{conv}}(W)$, the closure taken with respect to the topology of coordinatewise convergence in ℓ_∞.

PROBLEM 3.4. Suppose that $x \in c_{00}$ and $x_j = 0$ for every $j > n$. Show that

$$\|x\| = \|x\|_m \quad \text{for each } m \geqslant \left\lfloor \frac{n-1}{2} \right\rfloor,$$

where $(\|\cdot\|_m)_{m=0}^\infty$ is the increasing sequence of norms defining the norm on $\mathcal{T}$, given by the recursive formula:

$$\|x\|_0 = \|x\|_\infty \quad \text{and} \quad \|x\|_m = \max\left\{ \|x\|_{m-1},\ \frac{1}{2} \sup_{k \leqslant E_1 < \ldots < E_k} \sum_{j=1}^k \|E_j x\|_{m-1} \right\}.$$

PROBLEM 3.5. Prove that the sequence $(\|\cdot\|)_{m=1}^\infty$ of norms defining the norm on $\mathcal{T}$, and the canonical basis $(e_n)_{n=1}^\infty$ of $\mathcal{T}$, have the following elementary properties:

(a) For all strictly increasing sequences $(m_j)_{j=1}^\infty$ and $(n_j)_{j=1}^\infty$ of natural numbers with $m_j \leqslant n_j$ for each $j \in \mathbb{N}$, every sequence of scalars $(a_j)_{j=1}^N$, and every $m \in \mathbb{N}_0$, we have $\| \sum_{j=1}^N a_j e_{m_j} \|_m \leqslant \| \sum_{j=1}^N a_j e_{n_j} \|_m$.

(b) If a subsequence $(e_{n_j})_{j=1}^\infty$ of $(e_n)_{n=1}^\infty$ has a subsequence equivalent to $(e_n)_{n=1}^\infty$, then $(e_{n_j})_{j=1}^\infty$ is itself equivalent to $(e_n)_{n=1}^\infty$.

(c) For every $x \in \mathcal{T}$ and $m \in \mathbb{N}_0$, the norm $\|x\|_{m+1}$ is equal either to $\|x\|_0$, or $\frac{1}{2} \sup_{k \leqslant E_1 < \ldots < E_k} \sum_{j=1}^k \|E_j x\|_m$. In other words, unless the Tsirelson norm of x is just the supremum norm, at each step m we can pick the latter expression under the maximum sign defining $\|x\|_{m+1}$ (although the sequence of norms may stabilize—it actually always stabilizes for $x \in c_{00}$; cf. Problem 3.4).

PROBLEM 3.6. Given a basis $(e_n)_{n=1}^\infty$ of a Banach space X, we say that is *symmetric* (respectively: *subsymmetric*) provided that $(e_n)_{n=1}$ is equivalent to $(e_{\sigma(n)})_{n=1}^\infty$ for every permutation σ of $\mathbb{N}$ (respectively: is unconditional and equivalent to $(e_{n_j})_{j=1}^\infty$ for every strictly increasing sequence $(n_j)_{j=1}^\infty \subset \mathbb{N}$). Show that $\mathcal{T}$ contains no subsymmetric basic sequence.

Hint. There are many ℓ_1^n-like subspaces inside $\mathcal{T}$.

Remark. Note that symmetric bases are automatically unconditional, while we cannot drop the requirement of unconditionality in the definition of subsymmetric bases (as is shown by the summing basis of c). As a small side exercise you may like to prove that every symmetric basis is subsymmetric but the converse is not true. For the latter statement, consider Garling's example: X is the space of all real sequences $(a_n)_{n=1}^\infty$ such that

$$\|(a_n)_{n=1}^\infty\| := \sup\left\{ \sum_{k=1}^\infty k^{-1/2} |a_{n_k}| \colon (n_k)_{k=1}^\infty \nearrow \infty \right\} < \infty.$$

PROBLEM 3.7. Show that the Tsirelson space with parameter $\theta = N^{-1}$, for a natural number N, is $(N - \varepsilon)$-distortable for every $\varepsilon > 0$, that is, there exists an equivalent norm $|\cdot|$ on $\mathcal{T}(\mathcal{S}, N^{-1})$ such that

$$\inf_{\substack{Y < \mathcal{T}(\mathcal{S}, N^{-1}) \\ \dim Y = \infty}} \sup\left\{ \frac{|x|}{|y|} \colon x, y \in Y,\ \|x\| = \|y\| = 1 \right\} \geqslant N - \varepsilon.$$

Hint. Define a new norm on $\mathcal{T}(\mathcal{S}, N^{-1})$ by $|x| = \sup\{\sum_{j=1}^{N} \|E_j x\| \colon E_1 < \ldots < E_N\}$ and consider two types on vectors (appearing in every infinite-dimensional subspace):

- the average of block sequences $(y_k)_{k=1}^n$ spanning almost isometric copies of ℓ_1^n (with n much larger than N), and
- the average of sequences $z_1 < \ldots < z_n$, where each z_j is an "$\ell_1^{n_j}$-average" as above with n_j's rapidly increasing.

PROBLEM 3.8. Show that for each $k \in \mathbb{N}$ the canonical basis $(e_n)_{n=1}^\infty$ of $\mathcal{T}$ is equivalent to $(e_{n+k})_{n=1}^\infty$ and, if we let C_k to be the minimal equivalence constant, then $\sup_k C_k = \infty$. Prove also that, similarly, $(e_n)_{n=1}^\infty \sim (e_{kn})_{n=1}^\infty$ for each $k \in \mathbb{N}$ and that the corresponding equivalence constants grow to infinity as $k \to \infty$.

PROBLEM 3.9. For each $m \in \mathbb{N}_0$ define a norm on c_{00} by the formula

$$\|x\|_m' = \sup\left\{\sum_{j=1}^{3} \|E_j x\|_m \colon 1 \leqslant E_1 < E_2 < E_3\right\},$$

where the inside norm is the mth norm in the sequence defining Tsirelson's norm. Show that for every strictly increasing sequence $(n_j)_{j=1}^\infty \subset \mathbb{N}$, every finite sequence of scalars $(a_j)_{j=1}^N$ and every $m \in \mathbb{N}_0$ we have

$$\left\|\sum_{j=1}^{N} a_j e_{n_{2j}}\right\|_m \leqslant \left\|\sum_{j=1}^{N} a_j e_{n_j}\right\|_m' .$$

PROBLEM 3.10. Prove that every subsequence of the canonical basis of $\mathcal{T}$ spans a subspace which is isomorphic to its square. In particular, $\mathcal{T} \cong \mathcal{T} \oplus \mathcal{T}$.

Hint. Use the assertions of Problems 3.5(a) and 3.9.

Remark. This result, as well as the assertion of Problem 3.9 which is a crucial lemma here, comes from the paper [P.G. Casazza, W.B. Johnson, L. Tzafriri, *On Tsirelson's space*, Israel J. Math. **47** (1984), 81–98].

PROBLEM 3.11. Let $(e_n)_{n=1}^\infty$ be the canonical basis of $\mathcal{T}$. Show that for every strictly increasing sequence $(n_j)_{j=1}^\infty \subset \mathbb{N}$, the basic sequence $(e_k)_{k \neq n_{2j}}$ is 3-equivalent to $(e_k)_{k=1}^\infty$.

Hint. Use the assertions of Problems 3.5(a) and 3.9.

PROBLEM 3.12. Let $(e_n)_{n=1}^\infty$ be the canonical basis of $\mathcal{T}$. Show that for each $k \in \mathbb{N}$ and every sequence of scalars $(a_j)_{j=1}^N$ we have

$$\left\|\sum_{n=1}^{N} a_n e_{n+k-1}\right\| \leqslant \left\|\sum_{n=1}^{N} a_n e_{kn}\right\| \leqslant 4\left\|\sum_{n=1}^{N} a_n e_{n+k-1}\right\|.$$

Hint. Apply repeatedly the assertion of Problem 3.9 to the sequence given by $x = \sum_{n=1}^N a_n e_{2^j \cdot n}$, where j is chosen so that $2^{j-1} \leqslant k < 2^j$.

Remark. The above assertion gives a kind of compromise between equivalences formulated in Problem 3.8 and unboundedness of the equivalence constants. It implies, e.g., that $(e_n)_{n=k+1}^\infty$ and $(e_{2^k \cdot n})_{n=1}^\infty$ are 16-equivalent.